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Mappings Related to Permutations

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For each permutation of an interval of integers, a mapping is constructed with certain properties. Even for the infinite intervals ℕ or ℤ, the result is constructive and easily implemented. A second algorithm is given and utilized to describe the resulting mapping for permutations of a particularly regular type. The question of existence of these mappings arises in proving that certain integral conditions imply various smoothness properties.

This paper presents a construction of a mapping with certain properties for each permutation of an interval of integers. Even for the infinite cases, the result is constructive and easily implemented. A second algorithm is given and utilized to describe the resulting mappings for permutations of a particular type.

Taylor [7] first proved the existence of these mappings for finite intervals using the Philip Hall Theorem on Systems of Distinct Representatives. A second proof, which did not use the Philip Hall theorem, was given by Garsia and Rodemich [2]. One of us found an extension of the result to infinite intervals using the König Infinity Lemma and the Cantor–Bernstein theorem. Here we shall present a modification of the argument of Garsia and Rodemich which, in contrast to theirs, generalizes directly to the cases of infinite intervals. In addition, our proofs shows how to construct the mapping from the permutation, so that a recursive permutation has a recursive mapping. This was not apparent from the proof using the König Infinity Lemma as is known from [5].

ORIGIN AND DESCRIPTION OF THE THEOREM

To motivate the combinatorial question, the original real analysis problem must be considered. In a successful effort to unify the theory of several

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smoothness results involving integral conditions, Garsia [1-3] first reduced a very general integral inequality to this form: If f is a real-valued, measurable function on [0, 1] and Φ is a nondecreasing function on $[0, +\infty)$, then for every δ, $0 < \delta < 1$,

$$\iint_{|x-y|\leqslant\delta} \Phi(|f^*(x) - f^*(y)|)\, dx\, dy \leqslant \iint_{|x-y|\leqslant\delta} \Phi(|f(x) - f(y)|)\, dx\, dy, \quad (1)$$

where the integration is understood to be over the subset of $[0, 1] \times [0, 1]$ for which $|x - y| \leqslant \delta$, and f^* is the nonincreasing rearrangement of f as described by Hardy, Littlewood, and Pólya [4]. The rearrangement of the domain of f is done in a measure-preserving way so that for any λ, the measure of $\{x: f^*(x) > \lambda\}$ is the same as the measure of $\{x: f(x) > \lambda\}$. The function f^* can be defined by

$$f^*(x) \equiv \inf\{\lambda: m\{t: f(t) > \lambda\} \leqslant x\}.$$

Garsia noted that the problem is equivalent to a combinatorial question. Consider the integral inequality (1) for continuous f. Discretize the integrals by considering a uniform partition $x_0 < x_1 < \cdots < x_n$ of [0, 1] into n intervals. Let $y_i = f^*(x_i)$. The value y_i is an approximation of f^* on the ith interval of the partition. Consider f as a rearrangement of f^* by noting there is some permutation Σ of $\{1,..., n\}$ such that $f(x_i) \doteq y_{\Sigma(i)}$. By the monotonicity of Φ, the integral inequality would follow immediately if the pairs of values $y_{\Sigma(i)}$ and $y_{\Sigma(j)}$ for which $|i - j| \leqslant n\delta$ could be put in one-to-one correspondence with the pairs y_i and y_j for which $|i - j| \leqslant n\delta$ in such a fashion that the segment between $y_{\Sigma(i)}$ and $y_{\Sigma(j)}$ contained the segment between y_i and y_j.

The existence of that correspondence was conjectured by Garsia and later proved by Taylor [7]. Taylor's lemma is the assertion of the Theorem stated below when $y_i = i$ for $i = 1,..., n$, $M = [n\delta]$, and $I = \{1,..., n\}$. Continuing Garsia's program Milne [6] has been able to apply Taylor's lemma to integrals in several dimensions.

We preface the statement of the Theorem with the introduction of some notation and geometric perspective. For integers i and j let $[\![i,j]\!] = \{k \in \mathbb{Z}: \min(i,j) \leqslant k \leqslant \max(i,j)\}$. Note that $|i - j|$ is one less than the cardinality of the set $[\![i,j]\!]$. For I any subset of $\mathbb{Z}$, including infinite intervals let

$$\mathscr{D}^I = \{[\![i,j]\!]: 1 \leqslant |i - j| \text{ and } i \in I \text{ and } j \in I\}.$$

For M any positive integer let

$$\mathscr{D}_M{}^I = \{[\![i,j]\!] \in \mathscr{D}^I: |i - j| \leqslant M\}.$$

The following geometric representation of $\mathscr{D}^{\mathbb{Z}}$ helps in understanding Taylor's lemma and its generalizations. Consider the upper half-plane with half-lines of slope plus one and minus one emanating from each integer point on the real line. The lattice points of these lines are in obvious correspondence with intervals that are elements of $\mathscr{D}^{\mathbb{Z}}$ (see Fig. 1). Under the correspondence, $\alpha \subseteq \beta$, i.e., α is a subset of β, just in case α is "below" β in the geometric and lattice theoretic sense.

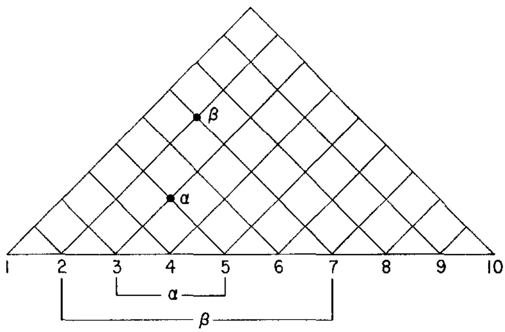

FIGURE 1

Let I be any interval of $\mathbb{Z}$, let M be a positive integer less than $|I|$, and let σ be a permutation of I. For each $\alpha = [\![i, j]\!] \in \mathscr{D}^I$, define $\sigma\alpha = [\![\sigma(i), \sigma(j)]\!]$. Further define $\sigma^{-1}\mathscr{D}_M{}^I = \{\sigma^{-1}\alpha \colon \alpha \in \mathscr{D}_M{}^I\}$, that is

$$\sigma^{-1}\mathscr{D}_M{}^I = \{[\![i, j]\!] \in \mathscr{D}^I \colon |\sigma(i) - \sigma(j)| \leqslant M\}.$$

THEOREM. *Let I be any interval of $\mathbb{Z}$, and let σ be any permutation of I. There exists a one-to-one correspondence π from $\mathscr{D}_M{}^I$ onto $\sigma^{-1}\mathscr{D}_M{}^I$ such that for each $\alpha \in \mathscr{D}_M{}^I$, $\alpha \subseteq \pi(\alpha)$.*

When I is finite, this is essentially Taylor's lemma, although stated for σ^{-1}. Observe that the map which takes each $\alpha \in \mathscr{D}_M{}^I$ to $\sigma^{-1}\alpha$ is a one-to-one correspondence between $\mathscr{D}_M{}^I$ and $\sigma^{-1}\mathscr{D}_M{}^I$. This map may not be increasing because there may exist an α which is not below $\sigma^{-1}(\alpha)$.

Our proof of the Theorem considers a sequence of approximations of σ on finite subintervals of I. We shall see how to construct, inductively, maps π_s for our approximations σ_s. Finally that construction will have enough stability to prove that the maps π_s converge to a map π with the desired properties.

Proof of the theorem. For each finite subinterval J of I define σ_J to be that permutation of J such that for each i and j in J, $\sigma_J(i) < \sigma_J(j)$ if and only if $\sigma(i) < \sigma(j)$. In computing examples, we find it instructive to work with $(\sigma_J)^{-1}$. Now suppose $J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots$, each J_s is a finite subinterval of I, and $\bigcup_{s=0}^{\infty} J_s = I$. Let $\sigma_s = \sigma_{J_s}$. If I is finite $\lim_{s\to\infty} \sigma_s(i) = \sigma(i)$ for each $i \in I$;

but if I is infinite $\lim_{s\to\infty} \sigma_s(i)$ may not exist (e.g., $\sigma(i) = -i$ for all i, $J_0 = \{0\}$, $J_{2s+1} = J_{2s} \cup \{s+1\}$, $J_{2s+2} = J_{2s+1} \cup \{-(s+1)\}$). However, for each i and j and s for which $[\![i,j]\!] \subseteq J_s$, $|\sigma_s(i) - \sigma_s(j)| \leqslant |\sigma_{s+1}(i) - \sigma_{s+1}(j)| \leqslant |\sigma(i) - \sigma(j)|$. Moreover, for each i and j, $\lim_{s\to\infty} |\sigma_s(i) - \sigma_s(j)| = |\sigma(i) - \sigma(j)|$. Indeed if $\sigma^{-1}(u) \in J_{s^*}$ for all $u \in [\![\sigma(i), \sigma(j)]\!]$, then $|\sigma_s(i) - \sigma_s(j)| = |\sigma(i) - \sigma(j)|$ for all $s \geqslant s^*$. Consequently, for any finite set of elements of $\sigma^{-1}\mathscr{D}_M{}^I$, for sufficiently large s, these elements are in $\sigma_s^{-1}\mathscr{D}_M^{J_s}$.

Now we turn to the construction of π. Suppose each J_s is a finite subinterval of I, $\bigcup_{s=0}^{\infty} J_s = I$, and $|J_0| = M+1$ while $|J_{s+1} - J_s| \leqslant 1$. The maps π_s for the permutations σ_s are defined inductively. For σ_0 notice that $\mathscr{D}_M^{J_0} = \sigma^{-1}\mathscr{D}_M^{J_0} = \mathscr{D}^{J_0}$ and let π_0 be the identity on $\mathscr{D}^{J_0}$. For J_{s+1} we may assume $|J_{s+1} - J_s| = 1$ and, by symmetry, that $\max(J_s) + 1 = b$ is the only element in $J_{s+1} - J_s$.

To aid in understanding the description of the construction, we shall follow through an example of the inductive step. Let $I = [\![1, 10]\!]$ and let σ be the permutation

$$\begin{array}{ccccccccccc} i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \sigma(i) & 7 & 8 & 2 & 9 & 3 & 5 & 1 & 10 & 4 & 6. \end{array}$$

It is most convenient to consider σ^{-1}, that is

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 3 & 5 & 9 & 6 & 10 & 1 & 2 & 4 & 8 \end{pmatrix}$$

or, more briefly, (7 3 5 9 6 10 1 2 4 8). Let $M = 2$, $s = 6$, and $J_6 = [\![1, 9]\!]$. Our definitions yield $(\sigma_6)^{-1} = (7\ 3\ 5\ 9\ 6\ 1\ 2\ 4\ 8)$ and $\sigma_7 = \sigma$. We represent π_6 in Fig. 2.

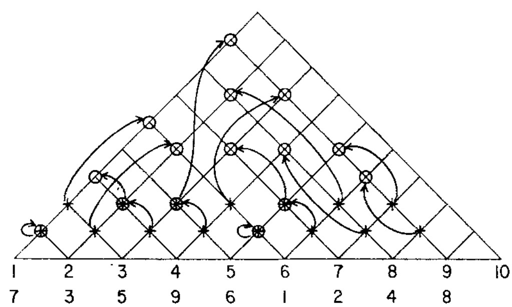

FIGURE 2

First consider those $[\![u, v]\!] \in \mathscr{D}_M^{J_s}$ for which $\pi_s([\![u, v]\!]) \in \sigma_{s+1}^{-1}\mathscr{D}_M^{J_{s+1}}$. Let $\pi_{s+1}([\![u, v]\!]) = \pi_s([\![u, v]\!])$ for each such $[\![u, v]\!]$. Next consider each $[\![u, v]\!] \in \mathscr{D}_M^{J_s}$ such that $\pi_s([\![u, v]\!])$ belongs to $\sigma_s^{-1}\mathscr{D}_M^{J_s} - \sigma_{s+1}^{-1}\mathscr{D}_M^{J_{s+1}}$. Let the interval $\pi_s([\![u, v]\!])$ be denoted by $[\![i, j]\!]$ with $i < j$. Since $[\![i, j]\!]$ belongs to $\sigma_s^{-1}\mathscr{D}_M^{J_s} - \sigma_{s+1}^{-1}\mathscr{D}_M^{J_{s+1}}$, we must have $|\sigma_s(i) - \sigma_s(j)| = M$ and $|\sigma_{s+1}(i) - \sigma_{s+1}(j)| = M + 1$. Thus $\sigma_{s+1}(b)$ must belong to $[\![\sigma_{s+1}(i), \sigma_{s+1}(j)]\!] - \{\sigma_{s+1}(i), \sigma_{s+1}(j)\}$, and hence both $|\sigma_{s+1}(i) - \sigma_{s+1}(b)| \leqslant M$ and $|\sigma_{s+1}(b) - \sigma_{s+1}(j)| \leqslant M$. Since $i < j < b$, $[\![i,j]\!] \subseteq [\![i, b]\!]$ and we can define $\pi_{s+1}([\![u, v]\!]) = [\![i, b]\!]$. Notice that $[\![u, v]\!] \subseteq \pi_s([\![u, v]\!]) = [\![i,j]\!] \subseteq [\![i, b]\!] = \pi_{s+1}([\![u, v]\!])$ as desired. Thus π_{s+1} has been defined on $\mathscr{D}_M^{J_s}$ and is one-to-one and increasing on $\mathscr{D}_M^{J_s}$. Continuing our example, the changes made in defining π_7 on $\mathscr{D}_2^{J_6}$ are represented in Fig. 3.

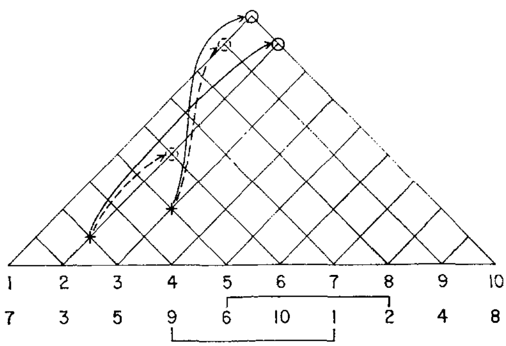

FIGURE 3

To complete our definition, we need only to define $\pi_{s+1}([\![i, b]\!])$ for $b - M \leqslant i < b$. Let R be the points in $\sigma_{s+1}^{-1}\mathscr{D}_M^{J_{s+1}}$ which are not yet in the range of π_{s+1} and hence are still available. Observe first that if $[\![i,j]\!] \in R$, then $[\![i,j]\!] \notin \mathscr{D}^{J_s}$; thus $\max(i,j) = b$. Next observe that $R = \sigma_{s+1}^{-1}\mathscr{D}_M^{J_{s+1}} - \pi_{s+1}\mathscr{D}_M^{J_s}$ so that $|R| = |\sigma_{s+1}^{-1}\mathscr{D}_M^{J_{s+1}}| - |\pi_{s+1}\mathscr{D}_M^{J_s}| = |\mathscr{D}_M^{J_{s+1}}| - |\mathscr{D}_M^{J_s}| = M$. Hence $R = \{[\![i_1, b]\!], [\![i_2, b]\!], \ldots, [\![i_M, b]\!]\}$ for some $i_1 > i_2 > \cdots > i_M$. It follows that $i_1 \leqslant b - 1$, $i_2 \leqslant b - 2, \ldots,$ and $i_M \leqslant b - M$. Thus for $1 \leqslant k \leqslant M$, $[\![b - k, b]\!] \subseteq [\![i_k, b]\!]$ and we may define $\pi_{s+1}[\![b - k, b]\!] = [\![i_k, b]\!]$.

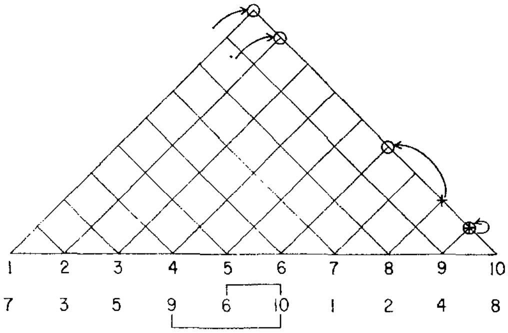

FIGURE 4

The results of the instructions in the preceding paragraph for our example are represented in Fig. 4. The appropriate combination of Figs. 2–4 is given in Fig. 5 and represents π_7, which is π.

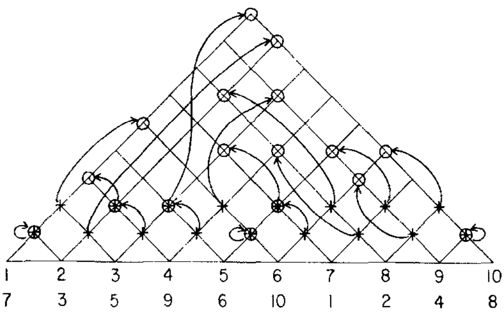

FIGURE 5

This completes our description of the construction of π_{s+1}. That π_{s+1} satisfies the conditions of the Theorem for σ_{s+1} on J_{s+1} is clear. Notice that if I is finite, then some $J_s = I$ so that this completes a proof of the Theorem for finite I. For infinite intervals I, we first show that for each $\alpha \in \mathscr{D}_M{}^I$, $\lim_{s\to\infty} \pi_s(\alpha)$ exists. It is then easy to prove that if we define $\pi(\alpha) = \lim_{s\to\infty} \pi_s(\alpha)$, π satisfies the conditions of the Theorem for σ.

Let $\alpha = [\![u, v]\!] \in \mathscr{D}_M{}^I$. We will now show that there are only finitely many s for which $\pi_{s+1}(\alpha) \neq \pi_s(\alpha)$. Let s^* be sufficiently large that $[\![u, v]\!] \subseteq J_{s^*}$, and consequently $\pi_s(\alpha)$ is defined for $s \geqslant s^*$. Let $i < j$ be such that $\pi_{s^*}(\alpha) = [\![i, j]\!]$. If $\pi_s(\alpha) \neq \pi_{s+1}(\alpha)$ for some $s > s^*$, let s_1 be the smallest such s. The inductive definition of π_s shows the existence of b_1 such that $\{b_1\} = J_{s_1+1} - J_{s_1}$ and $\sigma(b_1)$ is between $\sigma(i)$ and $\sigma(j)$, thus forcing $\sigma_{s+1}(i)$ and $\sigma_{s+1}(j)$ to be one unit farther apart than $\sigma_s(i)$ and $\sigma_s(j)$. For convenience assume $b_1 > j$. The value of $\pi_{s_1+1}(\alpha)$ is then defined to be $[\![i, b_1]\!]$. If there is a larger s such that $\pi_s(\alpha) \neq \pi_{s+1}(\alpha)$, let s_2 be the smallest such $s > s_1$. There must exist b_2 such that $\{b_2\} = J_{s_2+1} - J_{s_2}$ and $\sigma(b_2)$ is between $\sigma(i)$ and $\sigma(b_1)$. Consequently, $\sigma(b_2)$ is between $\sigma(i)$ and $\sigma(j)$ as was $\sigma(b_1)$. Continuing in this fashion, a sequence $b_1, b_2, \ldots$ of distinct integers is found such that $\sigma(b_n)$ is between $\sigma(i)$ and $\sigma(j)$ for $n = 1, 2, \ldots$. Since there are only $|\sigma(i) - \sigma(j)| - 1$ numbers between $\sigma(i)$ and $\sigma(j)$, $\pi_s(\alpha)$ cannot have more than that number of changes for $s \geqslant s^*$. Thus $\pi(\alpha) = \lim_{s\to\infty} \pi_s(\alpha)$ exists for each $\alpha \in \mathscr{D}_M{}^I$.

The map π just defined does map $\mathscr{D}_M{}^I$ into $\sigma^{-1}\mathscr{D}_M{}^I$. That $\alpha \subseteq \pi(\alpha)$ follows at once from the fact that for each s such that $\alpha \subseteq J_s$, $\alpha \subseteq \pi_s(\alpha)$. Similarly, π is one-to-one since each π_s is one-to-one. Finally, we must show that π is onto $\sigma^{-1}\mathscr{D}_M{}^I$. Let $[\![k, l]\!]$ belong to $\sigma^{-1}\mathscr{D}_M{}^I$, i.e., $|\sigma(k) - \sigma(l)| \leqslant M$; and let s^* be so large that $[\![k, l]\!] \subseteq J_{s^*}$. Then $|\sigma_{s^*}(k) - \sigma_{s^*}(l)| \leqslant |\sigma(k) - \sigma(l)| \leqslant M$, so that $[\![k, l]\!] \in \sigma^{-1}\mathscr{D}_M^{J_{s^*}}$. There must be an $\alpha \in \mathscr{D}_M^{J_{s^*}}$ such that $\pi_{s^*}(\alpha) = [\![k, l]\!]$.

For each $s \geqslant s^*$, $|\sigma_{s+1}(k) - \sigma_{s+1}(l)| \leqslant |\sigma(k) - \sigma(l)| \leqslant M$ so that $\pi_{s+1}(\alpha) = \pi_s(\alpha)$. Thus $[\![k, l]\!] = \pi_{s^*}(\alpha) = \lim_{s\to\infty} \pi_s(\alpha) = \pi(\alpha)$ and π is onto $\sigma^{-1}\mathscr{D}_M{}^I$. This completes the proof.

Algorithmic Considerations

The construction of π given in the proof computes π in $O(n)$ steps if I has cardinality n. If I is infinite and σ is recursive, the construction may be used to obtain a recursive π for σ since for any $\alpha \in \mathscr{D}_M$, $\pi(\alpha) = \pi_s(\alpha)$ for any s such that $\pi_s(\alpha) \in \sigma^{-1}\mathscr{D}_M$. In the case $I = \mathbb{N}$, any sequence J_s converging to I will produce a recursive map. For $I = \mathbb{Z}$ we need only choose the sequence of intervals J_s in such a way that if $J_{s+1} - J_s = \{M(s)\}$ then M is a recursive function.

The construction of $\pi\colon \mathscr{D}_M{}^I \to \sigma^{-1}\mathscr{D}_M{}^I$ in the proof depended upon the permutation σ. In contrast we now consider $\mathscr{R}$ as a given subset of $\mathscr{D}^I$. We shall see that if $\mathscr{R} = \sigma^{-1}\mathscr{D}_M{}^I$ for some σ then it is easy to construct such a σ. We then describe a construction which produces an increasing map $\pi\colon \mathscr{D}_M{}^I \to \mathscr{R}$ whenever such a map exists. Finally we shall see that if $\mathscr{R} = \sigma^{-1}\mathscr{D}_M{}^{\mathbb{Z}}$ and $\mathscr{R}$ is periodic, then there exists an increasing map π which is periodic in a sense with the same period.

In case $I = \mathbb{N}$, if $\mathscr{R} = \sigma^{-1}\mathscr{D}_M{}^{\mathbb{N}}$ for some permutation, then σ is uniquely determined. For every positive integer k, let $L_k = \{l\colon [\![k, l]\!] \in \sigma^{-1}\mathscr{D}_M{}^{\mathbb{N}}\}$. There is exactly one value k_1 such that L_{k_1} has only M elements. It follows that $\sigma(k_1) = 1$. There is exactly one value k_2 such that $k_2 \neq k_1$ and L_{k_2} has the maximal number of elements in common with L_{k_1}. It follows that $\sigma(k_2) = 2$. There is exactly one value k_3 such that $k_3 \neq k_1$ and $k_3 \neq k_2$ and L_{k_3} has the maximal number of elements in common with L_{k_2}. It follows that $\sigma(k_3) = 3$. Continuing in this fashion the permutation from which $\mathscr{R}$ arose can be recovered.

The situation is different for the other infinite case. If $\mathscr{R} = \sigma^{-1}\mathscr{D}_M{}^{\mathbb{Z}}$ for some permutation σ, then there are infinitely many permutations from which it may have arisen. For c any fixed integer consider the permutations $\sigma_1(k) \equiv \sigma(k) + c$ and $\sigma_2(k) \equiv \sigma(-k) + c$ for all $k \in \mathbb{Z}$. It is easily checked that $\mathscr{R} = \sigma_1^{-1}\mathscr{D}_M{}^{\mathbb{Z}}$ and $\mathscr{R} = \sigma_2^{-1}\mathscr{D}_M{}^{\mathbb{Z}}$. We find that permutation τ such that $\tau(0) = 0$ and $\tau^{-1}(-1) < \tau^{-1}(1)$ and $\mathscr{R} = \tau^{-1}\mathscr{D}_M{}^{\mathbb{Z}}$. There are exactly two values k_{-1} and k_1 for which L_k has the maximal number of elements in common with L_0. Suppose $k_{-1} < k_1$. Let $\tau(k_1) = 1$ and $\tau(k_{-1}) = -1$. Continue to define $\tau(k_2)$, $\tau(k_3)$,... as in the case $I = \mathbb{N}$. Also define $\tau(k_{-2})$,... in a similar fashion.

If $I = \{1, 2,..., n\}$ and $M < n$, and the set $\mathscr{R} = \sigma^{-1}\mathscr{D}_M{}^I$ for some permutation σ, there is exactly one other permutation that give rises to $\mathscr{R}$, namely, $\sigma'(k) \equiv \sigma(n + 1 - k)$ for $k \in I$. There are two integers for which L_k has minimal size. One of these integers is mapped into 1 by σ. The rest of the values of $\sigma(k)$ are determined as before by looking at the overlaps of the various sets L_k.

Now we turn to the problem of constructing a map $\pi\colon \mathscr{D}_M{}^I \to \mathscr{R}$ when $\mathscr{R}$ does not necessarily arise from a permutation. A general algorithm was developed by Bean and is presented here with his kind permission. Wik [8] has shown the existence of such mappings in certain cases where I is finite. An example derived from Wik's study and the mapping for it which is constructed by Bean's algorithm are presented in Fig. 6.

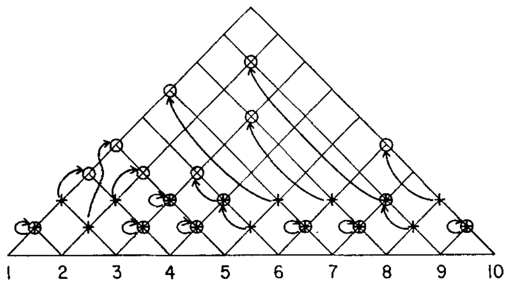

FIGURE 6

Bean's algorithm uses linear orderings of $\mathscr{D}_M{}^I$ and $\mathscr{R}$, which we now describe. Order the points of $\mathscr{D}_M{}^I$ by defining $[\![i,j]\!]$ to be "earlier" than $[\![i',j']\!]$ if $\min(i,j) < \min(i',j')$ or both $\min(i,j) = \min(i',j')$ and $\max(i,j) < \max(i',j')$. This ordering is lexicographic if each interval $[\![i,j]\!]$ is thought of as an ordered pair $(\min(i,j), \max(i,j))$. The elements of $\mathscr{R}$ are ordered by defining $[\![u, v]\!]$ to be "less" than $[\![u', v']\!]$ if $\max(u, v) < \max(u', v')$ or both $\max(u, v) = \max(u', v')$ and $\min(u, v) < \min(u', v')$. Figures 7 and 8 show the "earlier than" ordering of $\mathscr{D}_2{}^I$ and the "less than" ordering of $\mathscr{R} = \sigma^{-1}\mathscr{D}_2{}^I$ for our previous example.

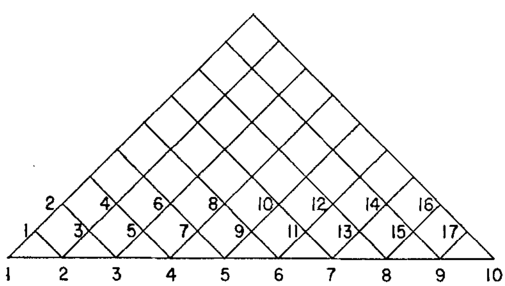

FIGURE 7

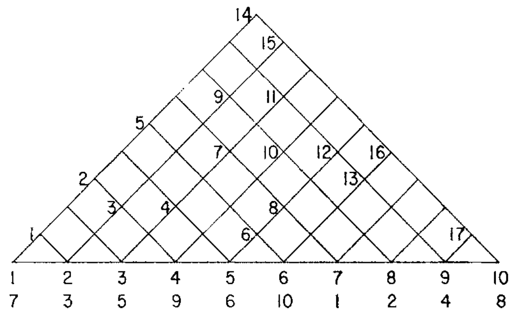

FIGURE 8

The following observation is the key to proving that the algorithm always succeeds.

EXCHANGE LEMMA. *Let $\mathscr{W}$ be a subset of $\mathscr{D}_M{}^I$. Suppose there exists a one-to-one, into function π: $\mathscr{W} \to \mathscr{R}$ which is increasing; i.e., if $\alpha \in \mathscr{D}_M{}^I$, then $\alpha \subseteq \pi(\alpha)$. Suppose further there exist α and α' in $\mathscr{D}_M{}^I$ such that α is earlier than α', $\pi(\alpha')$ is less than $\pi(\alpha)$, and α is below $\pi(\alpha')$. It follows that α' is below $\pi(\alpha)$; thus if the values assigned α and α' are exchanged, the new map is increasing. If π is onto, so is the new map.*

Let $\mathscr{W}$ be a subset of $\mathscr{D}_M{}^I$ with an earliest element. Let $\{\alpha_i\}_{0 \leqslant i < |\mathscr{W}|}$ be the enumeration of $\mathscr{W}$ in which $s < t$ implies α_s earlier than α_t. Suppose π is a one-to-one increasing map from $\mathscr{W}$ into $\mathscr{R}$. Suppose that for each $k \in I$ there exist only finitely many $l < k$ such that $[\![l, k]\!] \in \mathscr{R}$. Bean's algorithm, which we now describe, constructs an increasing, one-to-one map $\tilde{\pi}$ from $\mathscr{W}$ into $\mathscr{R}$.

The map is defined inductively. Let $\tilde{\pi}(\alpha_s)$ be the least element of $\mathscr{R}$ above α_s which is not in $\{\tilde{\pi}(\alpha_t)\}_{t<s}$. To see that there is always an interval available for α_s, we consider successive modifications of π. Let $\pi_0 = \pi$. If $\tilde{\pi}(\alpha_s) = \pi_s(\alpha)$ for some α later than α_s, then π_{s+1} is defined to be π_s except at that α and at α_s where the values are exchanged. Otherwise, π_{s+1} is to be π_s except at α_s where $\pi_{s+1}(\alpha_s) = \tilde{\pi}(\alpha_s)$. In the first case, the Exchange lemma shows that π_{s+1} is an increasing one-to-one map. Thus, $\pi_s(\alpha_s)$ is available for α_s when $\tilde{\pi}(\alpha_s)$ is defined. Note that if π is onto $\mathscr{R}$, $\tilde{\pi}$ is also.

Letting $\mathscr{W} = \mathscr{D}_M{}^I$ and applying the main theorem of this paper, we see that for any $I \neq \mathbb{Z}$, any permutation σ on I, and any M, this algorithm constructs a one-to-one increasing map π from $\mathscr{D}_M{}^I$ onto $\sigma^{-1}\mathscr{D}_M{}^I$. We do not know a direct proof of this fact. Our demonstration requires the Theorem, which gives the existence of a one-to-one, onto, increasing map π. In fact, the hypothesis that $\mathscr{R}$ arise from a permutation can be omitted if the existence of a map π is known for other reasons.

The Periodic Case

Finally we consider the case in which $I = \mathbb{Z}$, $\mathscr{R} = \sigma^{-1}\mathscr{D}_M{}^{\mathbb{Z}}$ for some permutation σ, and $\mathscr{R}$ is periodic with period p. The periodicity means if $[\![u, v]\!] \in \mathscr{R}$ then $[\![u + p, v + p]\!]$ and $[\![u - p, v - p]\!]$ belong to $\mathscr{R}$. Any permutation σ for which either $\sigma(i + p) = \sigma(i) + p$ for all i or $\sigma(i + p) = \sigma(i) - p$ for all i has a graph $\mathscr{R}$ with period p. The converse can be shown using the method of recovering σ from $\mathscr{R}$.

The Theorem shows the existence of a one-to-one, increasing map $\pi\colon \mathscr{D}_M{}^{\mathbb{Z}} \to \mathscr{R}$. Let $\hat{\pi}$ be the restriction of π to $\mathscr{D}_M{}^{\mathbb{N}}$. The map $\hat{\pi}$ is one-to-one and increasing from its domain into $\mathscr{R}$. By our discussion of Bean's algorithm, it constructs a one-to-one, increasing map $\tilde{\pi}$ from $\mathscr{D}_M{}^{\mathbb{N}}$ into $\mathscr{R}$.

Since $\mathscr{R}$ arises from a permutation, for each i there are at most $2M$ elements $[\![i, j]\!] \in \mathscr{R}$ with $j > i$. The finitely many elements $[\![i, j]\!]$ in $\mathscr{R}$ with $1 \leqslant i \leqslant p$ and $j > i$ completely determine $\mathscr{R}$ since $\mathscr{R}$ is periodic. It follows that there exists a uniform bound B on $| i - j |$ for all $[\![i, j]\!] \in \mathscr{R}$.

For intervals $\alpha = [\![i, j]\!]$ and integers t, we define $\alpha + t$ to be $[\![i + t, j + t]\!]$. It will be shown that for all sufficiently late values of α, $\tilde{\pi}$ is a periodic map in the sense that if $\tilde{\pi}(\alpha) = \beta$, then $\tilde{\pi}(\alpha + p) = \beta + p$. As a first step, we make the following observation. If $\tilde{\pi}(\alpha) = \beta$, then for some α' earlier than or equal to $\alpha + p$, $\tilde{\pi}(\alpha') = \beta + p$. To verify the observation, consider the earliest α for which it fails and examine why $\alpha + p$ does not take on the value $\beta + p$.

This property of $\tilde{\pi}$ is used now to compare the following sets. For $\alpha \in \mathscr{D}_M{}^{\mathbb{N}}$ and k a positive integer, let $\mathscr{R}_\alpha(k)$ be the set of candidates for $\tilde{\pi}(\alpha + kp)$; that is

$$\begin{aligned}\mathscr{R}_\alpha(k) = \{\beta \in \mathscr{R}\colon &\alpha + kp \text{ is below } \beta \text{ and } \beta \text{ is not in} \\ &\text{the range of } \tilde{\pi} \text{ on elements earlier than } \alpha + kp\}.\end{aligned}$$

The observation of the preceding paragraph proves that the number of elements in $\mathscr{R}_\alpha(k)$ for a fixed α is a nonincreasing function of k for $k = 1, 2, \ldots$. Since $\mathscr{R}_\alpha(1)$ is finite, there is a number $K(\alpha)$ such that for all $k > K(\alpha)$, $\beta \in \mathscr{R}_\alpha(k)$ if and only if $\beta + p \in \mathscr{R}_\alpha(k + 1)$. It follows that if $\tilde{\pi}(\alpha + kp) = \beta$ then $\tilde{\pi}(\alpha + (k + 1)\, p) = \beta + p$. By the periodicity of $\mathscr{R}$ and the regularity of the construction, $\tilde{\pi}(\alpha + p) = \tilde{\pi}(\alpha) + p$ for all α such that $[\![0 + K([\![0, 1]\!]), 1 + K([\![0, 1]\!])]\!]$ is earlier than α. There is no p', $0 < p' < p$, for which $\tilde{\pi}(\alpha + p') = \tilde{\pi}(\alpha) + p'$ for all sufficiently late α since $\mathscr{R}$ would then have a period less than p.

It has been shown that $\tilde{\pi}$ is a one-to-one, increasing map into $\mathscr{R}$. Furthermore, $\tilde{\pi}$ is periodic in the desired sense for late elements of $\mathscr{D}_M{}^{\mathbb{N}}$. Starting from the periodic part, $\tilde{\pi}$ can be modified on the earlier portion of $\mathscr{D}_M{}^{\mathbb{N}}$ and extended to the rest of $\mathscr{D}_M{}^{\mathbb{Z}}$ so that a mapping $\pi^*\colon \mathscr{D}_M{}^{\mathbb{Z}} \to \mathscr{R}$ is obtained which is one-to-one, increasing, and periodic in the desired sense.

Our first step in showing π^* is onto is to determine the number of elements in a period of $\mathscr{R}$. For each i let $\mathscr{F}_i = \{[\![i,j]\!] \in \mathscr{R}: i < j\}$ and $\mathscr{B}_i = \{[\![j,i]\!] \in \mathscr{R}: j < i\}$. Since $\mathscr{R}$ arises from a permutation, we see that for each i, $|\mathscr{F}_i \cup \mathscr{B}_i| = |\mathscr{F}_i| + |\mathscr{B}_i| = 2M$ and hence $\sum_{i=1}^{p} |\mathscr{F}_i| + |\mathscr{B}_i| = 2Mp$. Since $\mathscr{R}$ is periodic with period p, for every $\beta \in \mathscr{R}$ there is a unique integer t such that $\beta + tp \in \bigcup_{i=1}^{p} \mathscr{F}_i$. In particular, this is true for each $\beta \in \bigcup_{i=1}^{p} \mathscr{B}_i$ and shows $|\bigcup_{i=1}^{p} \mathscr{B}_i| = |\bigcup_{i=1}^{p} \mathscr{F}_i|$. Since the $\mathscr{B}_i$ are pairwise disjoint and the $\mathscr{F}_i$ are also pairwise disjoint, $\sum_{i=1}^{p} |\mathscr{B}_i| = \sum_{i=1}^{p} |\mathscr{F}_i|$. Finally we conclude

$$\left|\bigcup_{i=1}^{p} \mathscr{F}_i\right| = \sum_{i=1}^{p} |\mathscr{F}_i| = \tfrac{1}{2}\sum_{i=1}^{p} (|\mathscr{F}_i| + |\mathscr{B}_i|) = Mp.$$

For each positive integer t,

$$|\mathscr{D}_M^{[1,tp]}| = M \cdot tp - \sum_{j=1}^{M} j = M(tp) - (M(M+1)/2) = |\pi^*(\mathscr{D}_M^{[1,tp]})|$$

since π^* is one-to-one. Furthermore, for each positive integer t, we let $\mathscr{R}^{[1,tp]}$ denote the set of elements in $\mathscr{R}$ which are above at least one element in $\mathscr{D}_M^{[1,tp]}$. The set $\mathscr{R}^{[1,tp]}$ is the union of the set of intervals in $\mathscr{R}$ above $[\![1,2]\!]$ with $\bigcup_{i=1}^{tp+1} \mathscr{F}_i$. Since there are at most B^2 intervals in $\mathscr{R}$ above $[\![1,2]\!]$,

$$|\mathscr{R}^{[1,tp]}| \leqslant B^2 + \left|\bigcup_{i=1}^{tp} \mathscr{F}_i\right| = B^2 + t\left|\bigcup_{i=1}^{p} \mathscr{F}_i\right| = B^2 + tMp.$$

Thus

$$\begin{aligned} |\mathscr{R}^{[1,tp]} - \pi^*(\mathscr{D}_M^{[1,tp]})| &\leqslant |\mathscr{R}^{[1,tp]}| - |\pi^*(\mathscr{D}_M^{[1,pt]})| \\ &\leqslant B^2 + tMp - (M(tp) - (M(M+1)/2)) \\ &\leqslant B^2 + (M(M+1))/2. \end{aligned}$$

To see that π^* is onto, suppose not. There would exist a $\beta \in \mathscr{R} - \pi^*(\mathscr{D}_M{}^{\mathbf{Z}})$. Since both π^* and $\mathscr{R}$ are periodic with periodic with period p, $\beta + tp \in \mathscr{R} - \pi^*(\mathscr{D}_M{}^{\mathbf{Z}})$ for every integer t. In particular, for every s, $\bigcup_{i=ps+1}^{p(s+1)} \mathscr{F}_i$ contains an element of $\mathscr{R} - \pi^*(\mathscr{D}_M{}^{\mathbf{Z}})$. This gives $|\mathscr{R}^{[1,tp]} - \pi^*(\mathscr{D}_M^{[1,tp]})| \geqslant t$ for every positive integer t. For any $t > B^2 + (M(M+1))/2$ this inequality contradicts the upper bound on $|\mathscr{R}^{[1,tp]} - \pi^*(\mathscr{D}_M^{[1,tp]})|$ established in the preceding paragraph. This proves that π^* is onto.

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